Some Generalizations of Alliances and Secure Sets in Graphs

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Some Generalizations of Alliances and Secure Sets in Graphs

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1 Introduction

Graph theory has a fascinating history that spans centuries. From its humble beginnings with Euler's seven bridge problem, it has grown into a powerful mathematical discipline with applications in many fields.

A graph $G = (V, E)$ is an ordered pair of a nonempty set V and a set E of unordered pairs of elements of V. The sets V and E are called vertex set and edge set of G respectively. The elements of V and E are vertices of G and edges of G respectively. The edge corresponding to vertices u and v is denoted by uv. The number of elements in V is the order of G, denoted by $o(G)$ and that of E is the size of G. If H is any graph, then $V(H)$ and $E(H)$ denote vertex set of H and edge set of H respectively. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let G be a graph and v be a vertex of G. If $uv \in E$ for some $u \in V$, then u is said to be adjacent to v. The open neighborhood $N(v)$ of v is $\{u \in V : uv \in E\}$. The number of elements in $N(v)$ is the degree of v, denoted by $deg(v)$. The closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The minimum and the maximum of degrees taken over all the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. For any $S \subseteq V$, $N(S) = \underset{s \in S}{\cup} N(s)$ is the open neighborhood of S and $N[S] = N(S) \cup S$ is the closed neighborhood of S. For any integer $k \geq 1$, the open k-neighborhood of v is $N_k(v) = \{u \in V : 0 < d(u, v) \leq k\}$. The closed k-neighborhood of v is $N_k[v] = \{u \in V : d(u, v) \leq k\}$. If $u \in N_k(v)$, then u is called a k-neighbor of v. For any $S \subseteq V$, $N_k(S) = \underset{s \in S}{\cup} N_k(s)$ and $N_k[S] = N_k(S) \cup S$ are the open neighborhood of S and the closed neighborhood of S respectively.

In 2001, P. Kristiansen et al. introduced alliances[36] in graphs. In a graph $G = (V, E)$, for a non empty subset S of V and for any vertex v in S, the vertex v itself and the neighbors of v lying in S are the defenders of v whereas the neighbors of v lying outside S are the attackers of v with respect to S (i. e. For a vertex v in an alliance S, $N[v] \cap S$ is the set of all defenders whereas $N[v] \cap (V - S)$ is the set of all attackers of v with respect to S). Comparing the cardinalities of $N[v] \cap S$ and $N[v] \cap (V - S)$ by varying v over various subsets of $N[S]$, several alliances were introduced from 2001 to till date.

As a generalization of defensive alliances, secure sets were introduced by R.C. Brigham et al. in 2007. An attack on a set of vertices is a collection of mutually disjoint sets of attackers of respective vertices whereas a defense is that of their defenders. In 2012, fractional secure sets are introduced by G. Isaak et al. as a further generalization. Till date, several varieties of secure sets are introduced and their properties have been studied.

In this thesis, these concepts are generalized in various directions and studied their properties.

2 Objectives

The major objectives are as follows.

1. Generalization of secure sets by considering distant attackers/defenders for a vertex.

2. To define k-secure sets as an extension of secure sets.

3. To study ultra security in the context of generalized secure sets.

4. To study k-strong defensive alliances.

5. To define k-strong secure sets and study their properties.

3 Outline of the thesis

The thesis is divided into following five chapters.

Preliminaries
Chapter 1 : Alliances and Secure Sets
Chapter 2 : Distance Secure Sets
Chapter 3 : Generalized Distance Secure Sets
Chapter 4 : k-Secure Sets
Chapter 5 : Strong Defensive Alliances and Strong Secure Sets
Conclusion
Appendix
Bibliography

3.1 Preliminaries

This chapter contains basic notions of graph theory, which are used in the thesis. The notions not defined here are found in [9, 52].

Definition 1. *Two graphs G and H are **isomorphic** to each other if there is a bijection $f : V(G) \to V(H)$ such that for any distinct $u, v \in V(G)$, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If G is isomorphic to H, it is denoted by $G \cong H$.*

If G and H are isomorphic, they can be treated as a single graph though $V(G) \neq V(H)$. For an integer $n \geq 2$, a **path graph** (or simply, a path) P_n is a graph with $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. For an integer $n \geq 3$, a **cycle graph** (or simply a cycle) C_n is a graph with $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{\{i, i+1\}, \{1, n\} : 1 \leq i \leq n-1\}$.

Definition 2. *Let G be a graph and k be any positive integer. The k^{th} power of G, denoted by G^k, is the graph with vertex set $V(G)$ such that any two vertices are adjacent in G^k whenever the distance between them in G is less than or equal to k.*

Definition 3. *A dominating set in G is a set $S \subseteq V$ such that $N[S] = V$. The cardinality of a minimum dominating set is the domination number of G, denoted by $\gamma(G)$.*

Definition 4. *In a graph G, a set $S \subseteq V(G)$ is an independent set if the vertices of S are mutually non adjacent. For $l \geq 2$, an l-**partite graph** is a graph such that $V(G)$ can be partitioned into l independent sets.*

A 2-partite graph is called a bipartite graph.

Definition 5. *A graph G is a **complete graph** if its vertices are mutually adjacent. A complete graph of order n is denoted by K_n.*

Every graph of order n is a subgraph of K_n.

Definition 6. *The **complement** of a graph G, denoted by $\overline{G}$ is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_{o(G)}) - E(G)$.*

Definition 7. *The Cartesian product $G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is the graph with vertex set $V(G_1 \Box G_2) = V_1 \times V_2$ and edge set $E(G_1 \Box G_2) = \{(u, v)(x, y) : u = x$ and $vy \in E_2$, or $v = y$ and $ux \in E_1\}$.*

The Cartesian products of paths and cycles are called **grid like graphs**.

Definition 8. *The strong product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph K with vertex set $V_1 \times V_2 = \{(a, b) : a \in U, b \in V\}$ and two vertices (a, b) and (c, d) in $V_1 \times V_2$ are adjacent in K if and only if either $(a = c$ and $bd \in E_2), (b = d$ and $ac \in E_1)$ or $(ac \in E_1$ and $bd \in E_2)$.*

Definition 9. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The join $G_1 \vee G_2$ of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.*

Definition 10. *For $l \geq 2$, a **complete l-partite graph** is a graph of the form $\overline{K_{m_1}} \vee \overline{K_{m_2}} \vee \ldots \vee \overline{K_{m_l}}$ for some positive integers $m_1, m_2, \ldots, m_l$. It is denoted by $K_{m_1, m_2, \ldots, m_l}$.*

Definition 11. *A **wheel graph** $W_{1,n}$ is a graph of the form $\overline{K_1} \vee C_n$ for some integer $n \geq 3$. The vertices corresponding to C_n are called the rim vertices.*

3.2 Chapter 1: Alliances and Secure Sets

This chapter contains basic concepts of alliances and secure sets in graphs.

3.2.1 Various Alliances in Graphs

Alliances were introduced by P. Kristiansen et al. in 2001. This section contains basic notions of various alliances and their interrelationships in graphs.

Definition 12. *Let $G = (V, E)$ be a graph. A non empty set $S \subseteq V$ is*

i) a defensive alliance[36] if $|N[v] \cap S| \geq |N[v] - S|$ for all $v \in S$.

ii) an offensive alliance[36] if $|N[v] \cap S| \geq |N[v] - S|$ for all $v \in \partial S$.

iii) a powerful alliance[3, 4, 36] if $|N[v] \cap S| \geq |N[v] - S|$ *for all* $v \in N[S]$.

Later, several types of alliances and their generalizations were introduced namely global defensive alliances[36], global offensive alliances[43], defensive k-alliances[45], offensive r-alliances[45], powerful k-alliances, k-strong defensive alliances[25], etc.

Definition 13. *[45] For* $-\Delta < k \leq \Delta$*, a non empty set* S *of vertices is said to be a defensive* k*-alliance if* $|N(v) \cap S| \geq |N(v) - S| + k$ *for all* $v \in S$*. For* $-\Delta + 2 < k \leq \Delta$*, a non empty set* S *of vertices is said to be an offensive* k*-alliance if* $|N(v) \cap S| \geq |N(v) - S| + k$ *for all* $v \in \partial S$*.*

Defensive (-1)-alliances coincide with defensive alliances. Offensive alliances are nothing but offensive 1-alliances. A defensive k-alliance which is not a defensive $(k+1)$-alliance is called an exact defensive k-alliance[7].

Definition 14. *[15] For* $1 - \Delta \leq k \leq \Delta - 2$*, a set* $S \subseteq V$ *is a powerful* k*-alliance if* S *is both defensive* k*-alliance and offensive* $(k+2)$*-alliance in* G*.*

In a graph G, the cardinalities of minimum defensive alliances, minimum offensive alliances, minimum powerful alliances are respectively the defensive alliance number $a_d(G)$, offensive alliance number $a_o(G)$, powerful alliance number $a_p(G)$. The defensive alliance number is sometimes denoted by $a(G)$. The corresponding k-alliance numbers are defined similarly.

Global alliances are introduced by combining alliances and dominating sets. A set $S \subseteq V$ is a global defensive/offensive/powerful alliance[36] if it is a dominating set as well as a defensive/offensive/powerful alliance. The alliance numbers corresponding to these were defined similarly as above. Results on these alliances, their variants and corresponding global alliances are found in [1, 2, 3, 4, 6, 8, 10, 11, 14, 15, 16, 17, 18, 21, 22, 23, 29, 35, 37, 38, 42, 43, 45, 46, 47, 48, 49, 53, 54, 55]. As special types of defensive alliances, neighborhood alliances and accurate alliances are introduced in [44, 51] and [24] respectively.

In [31], L. H. Jamieson and B. C. Dean introduced weighted generalizations of defensive/ offensive/powerful alliances by assigning non negative integers to every vertex. If all of these integers are 1, then weighted alliances reduce to the alliances in [36].

Definition 15. *[56] For an integer* k *with* $-\Delta \leq k \leq \Delta$*, a set* $S \subset V$ *is a boundary defensive* k*-alliance if for every* $v \in S$*,* $|N(v) \cap S| = |N(v) - S| + k$*. For an integer* k *with* $2 - \Delta \leq k \leq \Delta$*, a set* $S \subseteq V$ *is a boundary offensive* k*-alliance if for every* $v \in \partial S$*,* $|N(v) \cap S| = |N(v) - S| + k$*. For an integer* k *with* $2 - \Delta \leq k \leq \Delta - 2$*, a set* $S \subseteq V$ *is a boundary powerful* k*-alliance if* S *is a boundary defensive* k*-alliance as well as boundary offensive* $(k+2)$*-alliance.*

Results on boundary k-alliances and boundary k-alliance numbers are found in [50, 54, 56, 57].

In general, subsets/supersets of a defensive alliance need not be defensive alliances. In this perspective, k-strong defensive alliances were introduced by C. Hegde and B. Sooryanarayana in 2017. A k-strong defensive alliance S is a defensive alliance such that its every subset having at least $|S| - k$ elements, is a defensive alliance.

Definition 16. *[25] For $k \geq 0$, a defensive alliance S is said to be a k-strong defensive alliance if for any l with $1 \leq l \leq k$, $S - S_l$ is a defensive alliance, where S_l is any subset of S with l elements.*

Results on 1-strong defensive alliances can be found in [25]. Further, Chapter-5 contains few more results on k-strong defensive alliances.

3.2.2 Secure Sets

Secure sets were introduced by R.C. Brigham et al. in 2007 by imposing a constraint that a vertex can defend/attack only a single vertex at a time.

Definition 17. *[5] Let $S = \{s_1, s_2, ..., s_m\}$ be a set of vertices. An attack $\mathcal{A}$ on S is an ordered m-tuple $(A_1, A_2, ..., A_m)$ (sometimes denoted by $\{A_i : 1 \leq i \leq m\}$) of disjoint sets A_i's of vertices such that $A_i \subseteq N[s_i] - S \ \forall \ i$ with $1 \leq i \leq m$. A defense $\mathcal{D}$ of S is an ordered m-tuple $(D_1, D_2, ..., D_m)$ (sometimes denoted by $\{D_i : 1 \leq i \leq m\}$) of mutually disjoint sets D_i's such that $D_i \subseteq N[s_i] \cap S \ \forall \ i$ with $1 \leq i \leq m$. An attack $\mathcal{A} = (A_1, A_2, ..., A_m)$ on S is said to be defendable if there exists a defense $\mathcal{D} = (D_1, D_2, ..., D_m)$ of S such that $|A_i| \leq |D_i|$ $\forall \ i$ with $1 \leq i \leq m$. A non empty set S of vertices is said to be a secure set if every attack on S is defendable. Otherwise, S is an insecure set.*

In the above definition, the sets A_i's and D_i's are called as the attacking/defending sets of s_i with respect to the attack $\mathcal{A}$/defense $\mathcal{D}$. For any vertex s in a secure set S, by taking the attacking set of s to be $N[s] - S$ and all other attacking sets to be empty, one can easily observe that every secure set is a defensive alliance.

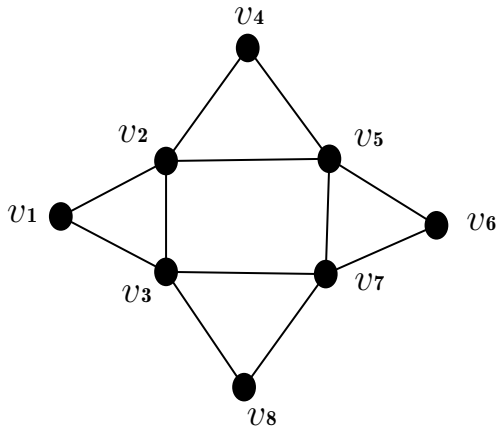

Figure 1: The graph G

In the graph G of Figure 1, consider the set $S = \{v_1, v_2, v_3, v_4\}$. The attacks on S are $\mathcal{A}_1 = (\phi, \phi, \phi, \phi)$, $\mathcal{A}_2 = (\phi, \phi, \phi, \{v_5\})$, $\mathcal{A}_3 = (\phi, \phi, \{v_7\}, \phi)$, $\mathcal{A}_4 = (\phi, \phi, \{v_7\}, \{v_5\})$, $\mathcal{A}_5 = (\phi, \phi, \{v_8\}, \{v_5\})$, $\mathcal{A}_6 = (\phi, \phi, \{v_8\}, \phi)$, $\mathcal{A}_7 = (\phi, \phi, \{v_7, v_8\}, \phi)$, $\mathcal{A}_8 = (\phi, \phi, \{v_7, v_8\}, \{v_5\})$, $\mathcal{A}_9 = (\phi, \{v_5\}, \{v_7\}, \phi)$, $\mathcal{A}_{10} = (\phi, \{v_5\}, \{v_8\}, \phi)$, $\mathcal{A}_{11} = (\phi, \{v_5\}, \{v_7, v_8\}, \phi)$, $\mathcal{A}_{12} = (\phi, \{v_5\}, \phi, \phi)$. Some of the defenses of S are $\mathcal{D}_1 = (\phi, \{v_1, v_2\}, \{v_3\}, \{v_4\})$, $\mathcal{D}_2 = (\phi, \{v_1, v_2, v_4\}, \{v_3\},$

$\phi)$, $\mathcal{D}_3 = (\phi, \{v_2\}, \{v_1, v_3\}, \{v_4\})$, $\mathcal{D}_4 = (\phi, \{v_2, v_4\}, \{v_1, v_3\}, \phi)$, $\mathcal{D}_5 = (\phi, \{v_1\}, \{v_3\}, \{v_2, v_4\})$, $\mathcal{D}_6 = (\phi, \{v_2, v_4\}, \{v_1, v_3\}, \phi)$, $\mathcal{D}_7 = (\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\})$. Here it is easy to observe that $\mathcal{D}_3$ defends S against any attack. Therefore S is a secure set.

In general, there need not exist single defense which defends the set against every attack. The set $S' = \{v_2, v_3, v_5, v_7\}$ is a secure set. However, there is no single defense of S' which defends S' against all the attacks on it. The following is a characterization of secure sets.

Theorem 3.1. *[5] A set S is a secure set if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$.*

An alternative proof of Theorem 3.1 can be found in [30] using the following result due to P. Hall.

Theorem 3.2. *(Hall's Theorem)[19, 20, 41]Suppose $A_1, A_2, ..., A_n$ are sets and $a_1, a_2, \ldots, a_n$ are non negative integers. There exist pairwise disjoint sets $B_1, B_2, ..., B_n$ such that $B_i \subseteq A_i$, $|B_i| = a_i$ for $1 \leq i \leq n$ if and only if for any $I \subseteq \{1, 2, ..., n\}$, $|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} a_i$.*

The cardinality of a minimum secure set in G is the security number[5] of G, denoted by $s(G)$. The maximum cardinality of a minimal secure set is the upper security number[13] of G, denoted by $S(G)$. More results on secure sets and security number can be found in [5, 12, 13, 26].

A secure set is a global secure set if it is also a dominating set. Results on global secure sets are found in [27, 28, 32, 33, 34].

3.2.3 Fractional Security and Ultra Security

In the concept of secure sets, a vertex defends/attacks at most one vertex at a time. In real situations, for instance, in a war, a nation may not use the entire military force to attack/defend someone. To study such scenario, fractional secure sets are useful. In this context attacks/defenses are viewed as functions. While defining an attack on a set S, for each edge joining a vertex of S with that of $V - S$, the numbers between 0 and 1 are assigned so that the sum of these numbers over a fixed vertex does not exceed 1 (Here, '1' can be seen as the full attacking capacity of a vertex). Similarly, a defense of S is defined by considering the edges of $\langle S \rangle$.

Definition 18. *[30] Let $\emptyset \neq S \subsetneq V$. An attack on S is a function $A : (V - S) \times S \to [0, 1]$ such that $A(u, v) = 0$ if $uv \notin E$ and $\sum_{v \in N(u) - S} A(u, v) \leq 1 \ \forall \ u \in V - S$. A defense of S is a function $D : S \times S \to [0, 1]$ such that for any $u, v \in S$, $D(u, v) = 0$ whenever $uv \notin E$ and $\sum_{v \in N[u] \cap S} D(u, v) \leq 1$. For any $v \in S$, denote $\sum_{u \in V - S} A(u, v) = A^*(v)$ and $\sum_{u \in S} D(v, u) = D^*(v)$.*

An attack A on S is said to defendable with a defense D of S if $A^*(v) \leq D^*(v)$ for all $v \in S$. An attack A/defense D is said to be an integer attack(defense) if the range of attack A/defense D is $\{0, 1\}$.

Definition 19. *[30] A set S is said to be an*
1. (I, I)-secure set if every integer attack on S is defendable with an integer defense.
2. (I, F)-secure set if every integer attack on S is defendable with a defense.
3. (F, F)-secure set if every attack on S is defendable with a defense.
4. (F, I)-secure set if every attack on S is defendable with an integer defense.

Remark 3.3. *(I, I)-secure sets coincide with secure sets defined in [5].*

Theorem 3.4. *[30] For $S \subseteq V$, the following are equivalent.*

1. *S is an (I, I)-secure set.*
2. *S is an (F, F)-secure set.*
3. *S is an (I, F)-secure set.*

Theorem 3.5. *[30] An (F, I)-secure set is an (I, I)-secure set.*

An (I, I)-secure set need not be an (F, I)-secure set. A characterization of (F, I) secure sets is obtained in [40, 39].

Let $G = (V, E)$ be a graph. For any $X \subseteq S \subseteq V$, let $E[X, N[X] - S]$ be the set of edges of G between X and $N[X] - S$. Let G^X be the subgraph of G given by $G^X = \Big(X \cup (N[X] - S), E[X, N[X] - S]\Big)$. The following is a characterization of (F, I) secure sets.

Theorem 3.6. *[40] A set S is (F, I)-secure if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S| + |X| - c(G^X)$ where $c(G^X)$ denote the number of components of G^X.*

In case of security and fractional security, the main task is to find a defense to defend the given set of vertices against an attack. One may think about secure sets having a single defense which is successful against any attack. Such secure sets were studied by C. Petrie in 2012. A set S is an *ultra secure set*[40] if there exists an integer defense of S which is successful against any integer attack on S. The following characterization of ultra secure sets is proved using Hall's theorem.

Theorem 3.7. *[40] A set S is an ultra secure set if and only if for any $X \subseteq S$, $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$.*

Theorem 3.8. *[40] Every ultra secure set is an (F, I)-secure set.*

An (F, I) secure set need not be an ultra secure set.

3.2.4 Equivalence of ultra secure sets

As there are 4 types of secure sets, one may think that each of them yield separate ultra securities. But it is not true! By definition, ultra (F, I) security implies ultra (F, F) security and ultra (I, I) security both of which implies ultra (I, F) security. The following proposition is useful to prove that an ultra (I, F) secure set is also an ultra (F, I) secure set.

Proposition 3.9. *Let $S \subseteq V$ and $A : (V - S) \times S \to [0, 1]$ be any attack on S. Then $A^*(u) \leq |N(u) - S|$.*

Theorem 3.10. *An ultra (I, F) secure set is also an ultra (F, I) secure set.*

Results relating security, (F, I) security and ultra security are found in [40].

3.3 Chapter 2: Distance Secure Sets

3.3.1 Introduction

In the concept of secure sets, only neighboring vertices are regarded as potential attackers or defenders for any given vertex in the set. In many practical situations, attackers or defenders are not always neighboring vertices, and the impact of their actions varies with distance. Consequently, there is a need to broaden the concept of secure sets to include attackers or defenders beyond immediate neighbors. This motivates to introduce the concept of distance secure sets. However, before extending secure sets to greater distances, it is crucial to address several fundamental questions. Though some of these observations seem straightforward, they must be carefully observed before generalizing secure sets to larger distances.

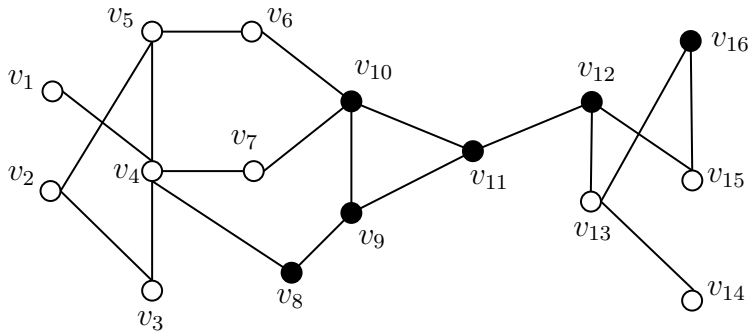

Figure 2: The graph G

In a graph $G = (V, E)$, let $S \subseteq V$ be a secure set and $s \in S$. Then we may observe the following.

1. The edges between s and the vertices of $N(s)$ can be seen as attacking/defending paths of s (as applicable).

2. There is only one attacking/defending path (which is the edge itself) between s and the vertices of $N(s)$. However this may not be the situation if we allow attackers/defenders from larger distances. There may even exist several paths of various lengths, which can be considered as attacking/defending paths. In the case when the lengths of attacking/defending paths are grater than 1, then intermediate vertices along the path also become significant. Further, these paths may enter/exit S several times.

   In the graph G of Figure 2, consider set S as set of all white colored vertices. Suppose we allow attackers and defenders to operate up to a distance of 4. Although vertex v_{10}

is within the 2-neighborhood of v_4, the shortest path from v_4 to v_{10} includes $v_7 \in S$ as an intermediate vertex, which may be considered as a repellent to the attacking path. In contrast, the path $v_{10}v_9v_8v_4$ does not include such a repellent vertex, but this path is of length 3. Similarly, for a defending path, even though $d(v_6, v_7) = 2$, the shortest path from v_6 to v_7 also contains an intermediate repellent for the defending path. Any path from v_6 to v_7 that avoids such a repellent is longer. This situation raises the question: "Which paths should be considered while defining attack/defense strategies?"

3. When the length of an attacking/defending path for a vertex exceeds 1, the intermediate vertices may also lie on the attacking or defending paths of other vertices, as these paths corresponding to different vertices can intersect.

   In the graph of Figure 2, if the path $v_{10}v_9v_8v_4$ serves as an attacking path of v_{10} on v_4, then the next question arises: "Do all vertices on this path attack on v_4?".

Further one can ask many such questions. While generalizing secure sets, addressing these type of inquiries may be difficult by solely relying on the k-neighborhoods of vertices of S within the original graph G. The following approach is helpful in this perspective.

3.3.2 Secure Sets: A Different Approach

In this section, secure sets are re-defined and the equivalence of the new definition and the definition given in [5] is established.

Let $S = \{s_i : 1 \leq i \leq m\} \subseteq V$. An attack $\mathcal{A} = \{A_i : 1 \leq i \leq m\}$ on S is a maximal attack if $\bigcup_{i=1}^{m} A_i = N[S] - S$. Note that an attack on S is maximal if every vertex of $N[S] - S$ is involved in the attack.

Proposition 3.11. *A set $S \subseteq V$ is a secure set if and only if every maximal attack on S is defendable.*

For any non empty set $S \subseteq V$, $Bord(S) = \{v \in S : N[v] \cap (V - S) \neq \emptyset\}$ is the border of S and $Int(S) = S - Bord(S)$ is the interior of S.

Proposition 3.12. *Let $\emptyset \subset S \subset V$. There exists a one to one correspondence between the maximal attacks on S and the functions $f : N[S] - S \to Bord(S)$ satisfying the condition that $f(x)$ is adjacent to x for all $x \in N[S] - S$.*

Theorem 3.13. *Let $S = \{s_i : 1 \leq i \leq m\} \subseteq V$ and $\mathcal{A} = \{A_i : 1 \leq i \leq m\}$ be an attack on S. Suppose a defense $\mathcal{D} = \{D_i : 1 \leq i \leq m\}$ is successful against $\mathcal{A}$. Then there exists a defense $\mathcal{C} = \{C_i : 1 \leq i \leq m\}$ of S with the following properties.*

1. *$\mathcal{C}$ is successful against $\mathcal{A}$.*
2. *For $1 \leq i \leq m$, $C_i = \emptyset$ whenever $s_i \in Int(S)$.*
3. *$\bigcup_{i=1}^{m} C_i = N[Bord(S)] \cap S$.*

4. For $1 \leq i \leq m$, $D_i \subseteq C_i$ whenever $s_i \in Bord(S)$.

Now it follows that to find a successful defense against an attack, it is enough if we consider the defenses with properties *2,3,4* of the above theorem. Similar to Proposition 3.12, it can be proved that the defenses of this type have one to one correspondence with the functions $D : N[Bord(S)] \cap S \to Bord(S)$ satisfying the condition that for any $x \in N[Bord(S)] \cap S$, $D(x)$ is equal or adjacent to x. Thus one may re-define secure sets as follows.

Definition 20. *Let $\emptyset \subset S \subset V$. An attack on S is a map $A : N[S] - S \to Bord(S)$ such that for any $x \in N[S] - S$, $A(x)$ is adjacent to x. A defense of S is a map $D : N[Bord(S)] \cap S \to Bord(S)$ such that for any $x \in N[Bord(S)] \cap S$, $D(x)$ is equal or adjacent to x. A set $S \subseteq V$ is a secure set if for any attack A on S, there is a defense D such that $|D^{-1}(z)| - |A^{-1}(z)| \geq 0$ $\forall$ $z \in Bord(S)$.*

In this new approach of secure sets, an attack and a defense on S is nothing but the assignment of neighbors to the vertices of S with some constraints. In the subsequent sections, this approach of secure sets is used for generalization in various ways.

3.3.3 k-Distance Secure Sets

Let k be an integer between 1 and $\text{diam}(G) = \max\{d(u, v) : u, v \in V\}$. In this section, k-distance secure sets are defined by generalizing Definition 20. Some properties of k-distance secure sets are discussed.

Definition 21. *Let $\emptyset \subset S \subset V$. A k-distance attack on S is a function $A : N_k[S] - S \to Bord(S)$ such that $d_H(x, A(x)) \leq k$ $\forall$ $x \in N_k[S] - S$, where $H = \langle (V - S) \cup \{A(x)\} \rangle$. Here x is said to be a k-distance attacker of $A(x)$ with respect to A. A k-distance defense of S is a function $D : N_k[Bord(S)] \cap S \to Bord(S)$ such that $d_{\langle S \rangle}\Big(x, D(x)\Big) \leq k$ $\forall$ $x \in N_k[Bord(S)] \cap S$. Here x is said to be a k-distance defender of $D(x)$ with respect to D.*

Remark 3.14.

1. *If A is a k-distance attack on S and if a vertex $x \in N_k[S] - S$ is a k-distance attacker of $y \in Bord(S)$, then there exists a path P between x and y with following properties.*
   - *(a) The length of P is at most k.*
   - *(b) All the vertices of P other than y belong to $V - S$.*
2. *If D is a k-distance defense of S, then for any $x \in S$, there exists a path between x and $D(x)$ of length at most k in $\langle S \rangle$.*

Example 3.15. *In the graph of Figure 2, define a map $A : N_3[S] - S \to Bord(S)$ by $A(v_{10}) = v_4, A(v_9) = v_{13}, A(v_{11} = v_6, A(v_8) = v_4, A(v_{12}) = v_{13}, A(v_{16}) = v_{15}$. Then A is a 3-distance attack on S. Here, though the distance between v_{10} and v_4 is 2, v_{10} is a 3-distance attacker as the shortest path from v_{10} to reach v_4 passing entirely through $V - S$, is of length 3. Similarly, v_6 can not be a 2-distance defender of v_7 as there is no $v_6 - v_7$ path of length at most 2, entirely lying inside S.*

Definition 22. *A k-distance attack A on S is defendable if there exists a k-distance defense D such that $|D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \geq |A^{-1}(z) \cap N_i^{\langle (V-S) \cup \{z\} \rangle}[z]| \; \forall \; i \in \{1, 2, \ldots, k\}$ and $z \in Bord(S)$. Here the defense D is called a successful defense of S against the attack A. A set $S \subseteq V$ is a k-distance secure set if every k-distance attack on S is defendable.*

Now, 1-distance secure sets coincide with secure sets of [5]. In simple words, a k-distance attack is defendable if there is a k-distance defense providing sufficient number of defenders at each distance up to k, for every vertex in the border of S.

Theorem 3.16. *Let G be a graph and k be an integer with $1 \leq k \leq \text{diam}(G)$.*

1. *For any $l \leq k$, if S is an l-distance secure set and if $N_k[S] - N_l[S] = \emptyset$, then S is a k-distance secure set.*
2. *Let S be a k-distance secure set. If $N_k[S] - N_{k-1}[S] \neq \emptyset$, then $|S| \geq k$.*
3. *If S is a k-distance secure set, then $|N_i[Bord(S)] \cap S| \geq |N_i[Bord(S)] - S| \; \forall \; i$, $1 \leq i \leq k$.*

Analogous to Theorem 3.1, a characterization of k-distance secure sets is derived as follows.

Theorem 3.17. *A set $S \subseteq V$ is a k-distance secure set if and only if for any component $\langle C \rangle$ of $\langle S \rangle$ and for any $X \subseteq Bord(C)$, $|N_i^{\langle C \rangle}[X]| \geq |N_i^{\langle (V-S) \cup C \rangle_C}[X] - S| \; \forall \; i$, $1 \leq i \leq k$.*

Corollary 3.18. *A connected set $S \subseteq V$ is a k-distance secure set if and only if for any $X \subseteq Bord(S)$, $|N_i^{\langle S \rangle}[X]| \geq |N_i^{G_S}[X] - S| \; \forall \; i$, $1 \leq i \leq k$.*

Corollary 3.19. *For any connected graph G of order $n \geq 4$, $ds_k(G) \leq n - 2 \; \forall \; k$.*

3.3.4 k-Distance Ultra Secure Sets

A special type of secure sets is introduced in [40]. A set $S \subseteq V$ is an ultra secure set [40] if a single defense of S defends S against any attack. The notion of ultra secure sets can be extended to k-distance secure sets.

Definition 23. *A set $S \subseteq V$ is said to be a k-distance ultra secure set if there exists a k-distance defense of S which defends S against every k-distance attack on S.*

A characterization of k-distance ultra secure sets
Let S be any subset of V and $Bord(S) = \{b_1, b_2, \ldots, b_s\}$. To derive a characterization of a k-distance ultra secure set, we construct a graph $G_k(S)$ as below.

1. Let $i = 1$, $j = 1$. Set $H = \langle S \rangle$ and $x = b_1$.
2. Attach $\langle N_k^{\langle (V-S) \cup \{x\} \rangle}(x) \rangle$ to H at x. Let H' be the resulting graph. Set $i = j$ and go to Step-3.
3. If $1 \leq i < s$, then set $j = i + 1$. Then set $H = H'$, $x = b_j$ and go to Step-2. Stop the process if $i = s$.

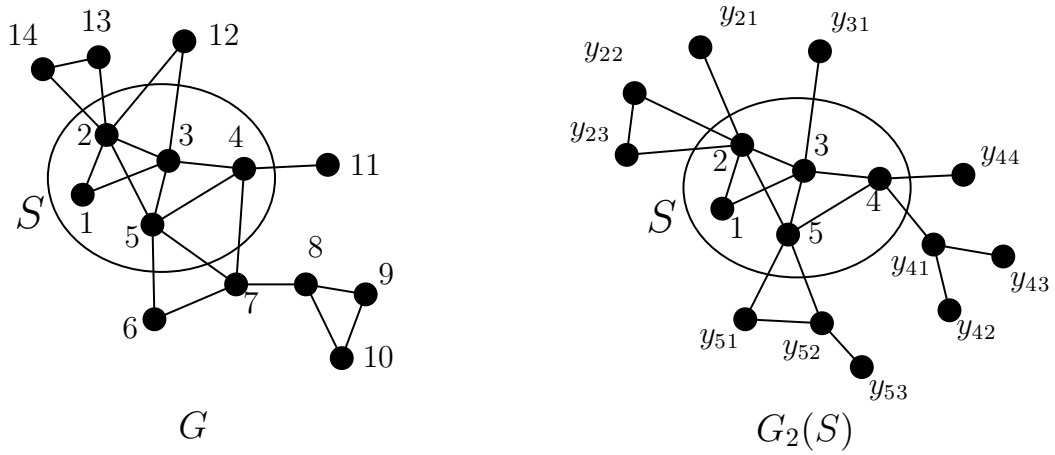

Figure 3: Construction of $G_2(S)$ from graph G with $S = \{1, 2, 3, 4, 5\}$.

An example for the construction of $G_k(S)$ is given in Figure 3.

Lemma 3.20. *For any $S \subseteq V$, there exists a unique k-distance attack A_S on S in $G_k(S)$. Further S is a k-distance ultra secure set in G if and only if A_S is defendable in $G_k(S)$.*

Lemma 3.21. *For any $S \subseteq V$, A_S is defendable in $G_k(S)$ if and only if for every component $\langle C \rangle$ of $\langle S \rangle$ and every $X \subseteq Bord(C)$, $|N_i^{\langle C \rangle}[X]| \geq \sum\limits_{x \in X} |N_i^{\langle (V-S) \cup \{x\} \rangle}[z] - S|$ for all i, $1 \leq i \leq k$ holds in G.*

Now Lemma 3.20 and Lemma 3.21 lead to the following characterization of k-distance ultra secure sets.

Theorem 3.22. *A set $S \subseteq V$ is k-distance ultra secure if and only if for every component $\langle C \rangle$ of $\langle S \rangle$ and for any $X \subseteq Bord(C)$, $|N_i^{\langle C \rangle}[X]| \geq \sum\limits_{x \in X} |N_i^{\langle (V-S) \cup \{x\} \rangle}[x] - S| \ \forall \ i$, $1 \leq i \leq k$.*

3.3.5 k-Distance Security Number

In this section, k-distance security numbers are obtained for certain classes of graphs.

Theorem 3.23. *For positive integers n, k with $n > 1$ and $1 \leq k \leq n-1$;*

$$ds_k(P_n) = \begin{cases} k & ; 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \\ \left\lceil \frac{n}{2} \right\rceil & ; k > \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

Theorem 3.24. *For integers $n \geq 3$ and $k \geq 1$;*

$$ds_k(C_n) = \begin{cases} 2k & ; 1 \leq k \leq \left\lfloor \frac{n}{4} \right\rfloor \\ \left\lceil \frac{n}{2} \right\rceil & ; k > \left\lfloor \frac{n}{4} \right\rfloor. \end{cases}$$

Theorem 3.25. *Let G be a non trivial graph with diameter k. Then $ds_k(G) \geq \left\lceil \frac{n}{2} \right\rceil$.*

Theorem 3.26. *For complete l-partite graph $G = K_{m_1,m_2,\ldots,m_l}$; $ds_2(G) = \left\lceil \frac{m_1+m_2+\cdots+m_l}{2} \right\rceil$ where $l \geq 2$.*

Theorem 3.27. *For $m, n \geq 2$, $ds_k(P_m \times P_n) \leq \{km, kn\}$.*

Theorem 3.28. *For integers $m \geq 3, n \geq 2$ and $k \geq 1$, $ds_k(C_m \times P_n) \leq min\{km, 2kn\}$.*

Theorem 3.29. *For integers $m, n \geq 3$ with $max\{m, n\} \geq 4$ and $k \geq 1$, $ds_k(C_m \times C_n) \leq min\{2kn, 2km\}$.*

It is easy to observe that $ds_2(P_2 \times P_4) = 4$, $ds_2(P_2 \times C_{25}) = 8$ and $ds_2(C_3 \times C_9) = 8$, this implies that the bounds obtained in Theorem 3.27, Theorem 3.28, Theorem 3.29 are strict.

3.4 Chapter 3: Generalized Distance Secure Sets

In this section, some more types of distance secure sets are introduced and their inter-relationships are studied. As discussed in the introduction of this chapter, there are several points to be observed while extending secure sets to a greater distance. By taking the conditions specified in Remark 3.14 into account, k-distance attacks and k-distance defenses are defined. A natural query arises about how the inclusion/exclusion of these conditions impacts the security of a set. This results in following types of attacks and defenses, the combinations of which give rise to four distinct types of distance-secure sets.

Definition 24. *A general k-distance attack on S is a function $A : N_k[S] - S \to S$ such that $d(x, A(x)) \leq k \ \forall \ x \in N_k[S] - S$.*
A general k-distance defense of S is function $D : S \to S$ such that $d(x, D(x)) \leq k \ \forall \ x \in S$.
A pure k-distance attack on S is a function $A : N_k[S] - S \to Bord(S)$ such that $d_H(x, A(x)) \leq k \ \forall \ x \in N_k[S] - S$, where $H = \langle (V - S) \cup \{A(x)\} \rangle$.
A pure k-distance defense of S is a function $D : S \to S$ such that $d_{\langle S \rangle}\Big(x, D(x)\Big) \leq k \ \forall \ x \in S$.

Remark 3.30. *A pure k-distance attack coincides with a k-distance attack defined earlier. A pure k-distance defense coincides with a $k-$distance defense when the domain and codomains are restricted to $N_k[Bord(S)] \cap S$ and $Bord(S)$ respectively.*

The defendability conditions for various combinations of attacks and defenses are as follows.

A general k-distance attack A on S is defendable by a general k-distance defense D of S if for every $z \in A(N_k[S] - S)$, $|D^{-1}(z) \cap N_i[z]| \geq |A^{-1}(z) \cap N_i[z]| \ \forall \ i, \ 1 \leq i \leq k$. A general k-distance attack A of S is defendable by a pure k-distance defense D of S if for every $z \in A(N_k[S] - S)$, $|D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \geq |A^{-1}(z) \cap N_i[z]| \ \forall \ i, \ 1 \leq i \leq k$. A pure k-distance attack A of S is defendable by a general k-distance defense D of S if for every $z \in Bord(S)$, $|D^{-1}(z) \cap N_i[z]| \geq |A^{-1}(z) \cap N_i^H[z]| \ \forall \ i, \ 1 \leq i \leq k$, where $H = \langle (V - S) \cup \{A(x)\} \rangle$. A pure k-distance attack A of S is defendable by a pure k-distance defense D of S if for every $z \in Bord(S)$, $|D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \geq |A^{-1}(z) \cap N_i^H[z]| \ \forall \ i, 1 \leq i \leq k$, where $H = \langle (V - S) \cup \{A(x)\} \rangle$.

Definition 25. *A non empty set $S \subseteq V$ is a*

1. *k-distance (G, G) secure set if every general k-distance attack on S is defendable by a general k-distance defense.*
2. *k-distance (G, P) secure set if every general k-distance attack on S is defendable by a pure k-distance defense.*
3. *k-distance (P, G) secure set if every pure k-distance attack on S is defendable by a general k-distance defense.*
4. *k-distance (P, P) secure set if every pure k-distance attack on S is defendable by a pure k-distance defense.*

Remark 3.31. 1. *For $k = 1$, all the above types of secure sets coincide with the secure sets defined in [5].*

2. *The definition of a k-distance (P, P) secure set coincides with a k-distance secure set defined in Definition 21.*

Theorem 3.32. *A set $S \subseteq V$ is a k-distance (G, G) secure set in G if and only if S is a secure set in G^i for all i with $1 \leq i \leq k$.*

Remark 3.33. 1. *k-distance (G, P) security $\implies$ k-distance (G, G) security $\implies$ k-distance (P, G) security.*

2. *k-distance (G, P) security $\implies$ k-distance (P, P) security $\implies$ k-distance (P, G) security.*

Theorem 3.34. *Every k-distance (P, G) secure set is a k-distance (G, G) secure set.*

Theorem 3.35. *Every k-distance (P, P) secure set is a k-distance (G, G) secure set.*

A k-distance (G, G) secure set need not be a k-distnace (P, P) secure set and a k-distance (P, P) secure set need not be a k-distance (G, P) secure set. In the graph of Figure 4, the set $S = \{1, 2, 5\}$ is a 2-distance (G, G) secure set, but since the attack A defined by $A(4) = A(3) = 5$ is not defendable by any pure 2-distance defense of S, S is not a 2-distance (P, P) secure set. In the graph of Figure 5, $\{1, 2, 3, 4, 7, 8, 9\}$ is a 3-distance (P, P) secure set; but not a 3-distance (G, P) secure set since the attack A given by $A(5) = A(6) = A(10) = A(11) = 7$ is not defendable by any pure 3-distance defense of S.

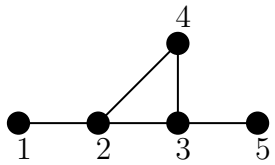

Figure 4

Theorem 3.17 characterizes k-distance (P, P) secure sets. The subsequent results are similar to Theorem 3.17.

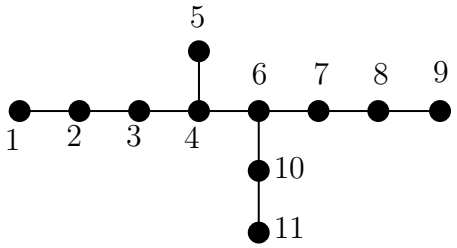

Figure 5

Theorem 3.36. *A set S is a k-distance (G, P) secure set if and only if for every $X \subseteq S$, $|N_i^{\langle S \rangle}[X]| \geq |N_i[X] - S| \ \forall \ i, \ 1 \leq i \leq k$.*

Theorem 3.37. *A set S is a k-distance (G, G) secure set if and only if for every $X \subseteq S$, $|N_i[X] \cap S| \geq |N_i[X] - S| \ \forall \ i, \ 1 \leq i \leq k$.*

Theorem 3.38. *A set S is a k-distance (P, G) secure set if and only if for every for every $X \subseteq Bord(S)$, $|N_i[X] \cap S| \geq |N_i^{\langle (V-S) \cup Bord(S) \rangle_{Bord(S)}}[X] - S| \ \forall \ i, 1 \leq i \leq k$.*

3.5 Chapter 4: k-Secure Sets

In many practical situations, it is required to have more stronger defense than each attack. In this direction, as an extension of defensive alliances, defensive k-alliances are introduced in [45] and its properties are discussed in [42]. For any integer $k \in \{-\Delta(G), \ldots, \Delta(G)\}$, a nonempty set $S \subseteq V$ is a defensive k-alliance[45] in G whenever $|N[x] \cap S| - |N[x] - S| \geq k$ for all $x \in S$. In this chapter, k-secure sets are introduced analogues to defensive k-alliances.

3.5.1 k-Secure Sets

In this section, k-secure sets are defined and some of their properties are discussed.

Definition 26. *For any integer k with $-\Delta(G) < k < \Delta(G)$, a set $S \subseteq V$ is a k-secure set if for any attack A on S, there exists a defense D of S such that $|D^{-1}(z)| - |A^{-1}(z)| \geq k \ \forall \ z \in Bord(S)$.*

The minimum cardinality of a k-secure set in G is the k-security number of G, denoted by $s_k(G)$. In this chapter, only the case $k \geq 0$ is considered. A 0-secure set is nothing but a secure set defined in [5]. The following theorem characterizes k-secure sets.

Theorem 3.39. *For any integer $k \geq 0$, a set $S \subseteq V$ is a k-secure set if and only if $|N[X] \cap S| \geq |N[X] - S| + k|X|$ for every $X \subseteq Bord(S)$.*

Corollary 3.40. *For any $k \geq 1$ and a k-secure set S, $Bord(S) \subsetneqq S$.*

The case $k = 0$ in Theorem 3.39 gives the following characterization of secure sets, which is also a sharpening of Theorem 3.1.

Theorem 3.41. *A set $S \subseteq V$ is a secure set if and only if $|N[X] \cap S| \geq |N[X] - S|$ for every $X \subseteq Bord(S)$.*

Proposition 3.42. *For any $k \geq 0$, if S is a minimal k-secure set, then $\langle S \rangle$ is connected.*

Theorem 3.43. *For $k \geq 0$, if S_1 and S_2 are k-secure sets with $S_1 \cap S_2 = \emptyset$, then so is $S_1 \cup S_2$.*

3.5.2 1-Secure Sets and Fractional Secure Sets

In this section, it is proved that 1-secure sets are also fractional secure sets. Characterizations of (F, I) secure sets and ultra secure sets are sharpened. The following lemma is useful to obtain a characterization of (F, I)-secure sets, which is a sharpening of Theorem 3.6

Lemma 3.44. *Let $X \subseteq S \subseteq V$. Then $|X| - c(G^X) = |X \cap Bord(S)| - c(G^{X \cap Bord(S)})$.*

The following theorem gives a characterization of an (F, I)-secure sets.

Theorem 3.45. *A set S is an (F, I)-secure set if and only if for every $X \subseteq Bord(S)$, $|N[X] \cap S| \geq |N[X] - S| + |X| - c(G^X)$.*

Theorem 3.46. *A 1-secure set is (F, I)-secure.*

Remark 3.47. *By Theorem 3.46, every 1-secure set is (F, I)-secure. By Theorem 3.8, every ultra secure set is also (F, I)-secure. However none of these two types imply the other in general.*

The following theorem is a sharpening of Theorem 3.7.

Theorem 3.48. *A set S is ultra secure if and only if $|N[X] \cap S| \geq \sum\limits_{x \in X} |N[x] - S|$ for every $X \subseteq Bord(S)$.*

Theorem 3.49. *If S is a 1-secure set and $G^{Bord(S)}$ has no cycle, then S is ultra secure.*

3.5.3 1-Security Number of Graphs

In this section, 1-security numbers of certain classes of graphs are obtained.

Theorem 3.50.

1. *$s_1(G) = 1$ if and only if $|V(G)| = 1$.*
2. *$s_1(G) = 2$ if and only if there exist two adjacent vertices u and v such that $deg(u) + deg(v) \leq 3$.*
3. *$s_1(G) = 3$ if and only if $s_1(G) \neq 2$ and there exists $S = \{u, v, w\} \subseteq V$ such that one of the following holds.*
   (a) $\langle S \rangle = P_3$ with $deg(u) = deg(v) = 1$ and $3 \leq deg(w) \leq 4$
   (b) $\langle S \rangle = P_3$ with $\langle S \cup \partial S \rangle = C_4$
   (c) $\langle S \rangle = C_3$ with $deg(u) = deg(v) = 2$ and $deg(w) \leq 4$.

Theorem 3.51.

1. *For any $n \geq 2$, $s_1(P_n) = 2$.*

2. $s_1(C_3) = s_1(C_4) = 3$. *For any* $n \geq 5$, $s_1(C_n) = 4$

3. *For any* n, $s_1(K_n) = n$.

4. *For* $n_1 \leq n_2 \leq ... \leq n_l$, $s_1(K_{n_1,n_2,...,n_l}) = n_1 + n_2 + ...n_{l-1} + \lceil \frac{n_l}{2} \rceil$

Theorem 3.52. 1. *For any* $m \leq n$ *with* $m \geq 2$, $s_1(P_m \times P_n) \leq 2m$.

2. *For any* $m \leq n$ *with* $m \geq 3$, $s_1(C_m \times C_n) \leq 4m$.

3. *For any* $m \geq 2$ *and* $n \geq 3$, $s_1(P_m \times C_n) \leq min\{4m, 2n\}$.

3.5.4 Ultra k-Secure Sets

In this section, ultra security and k-security are combined to define ultra k-secure sets.

Definition 27. *For* $0 \leq k < \Delta(G)$, *a defense* D *of* $S \subset V$ *is an ultra* k*-defense if for every attack* A *on* S, $|D^{-1}(z)| - |A^{-1}(z)| \geq k$ *for all* $z \in Bord(S)$. *A nonempty set* $S \subseteq V$ *is an ultra* k*-secure set if* $S = V$ *or* S *has an ultra* k*-defense.*

The following is a characterization of ultra k-secure sets.

Theorem 3.53. *For* $0 \leq k < \Delta(G)$, *a set* $S \subseteq V$ *is an ultra* k*-secure set if and only if* $|N[X] \cap S| \geq k|X| + \sum\limits_{x \in X} |N[x] - S|$ *for all* $X \subseteq Bord(S)$.

Theorem 3.54. *A necessary condition for a set* $S \subset V$ *to be an ultra* k*-secure set is* $|Bord(S)| \leq \frac{1}{k+1}|S|$.

Proposition 3.55. *For any graph* G, $k + 1 \leq s_k^u(G) \leq n$.

Theorem 3.56. *For a graph* G, $s_k^u(G) = k + 1$ *if and only if* $k = n - 1$ *or* G *has a block* B_1 *of* $k + 1$ *vertices and a block* B_2 *of two vertices having a common cut vertex of degree* $k + 1$ *such that* B_1 *is a pendant vertex in* $B(G)$, *which is adjacent to* B_2.

Theorem 3.57. *For any integer* $k \geq 1$, *if* $\delta(G) \geq n - k$, *then* $s_k^u(G) = n$.

Theorem 3.58. *For any integer* $k \geq 1$, *if* $\kappa \geq \frac{n}{k+1}$, *then* $s_k^u(G) = n$, *where* κ *is the vertex connectivity of* G.

Theorem 3.59. *For* $k \geq \frac{n-d_{n-1}}{d_{n-1}}$, $s_k^u(G) = n$, *where* d_{n-1} *denotes the number of vertices of degree* $n - 1$ *in* G.

3.5.5 Graphs with given Ultra k-Security Number

In [39, 40], it is shown that for each positive integer s, there exists a graph G of order $n > 2s$ such that $s(G) = s^u(G) = s$. In this section, for given positive integers n, k, s with $n \geq s \geq k + 1$, the construction of graphs of order n having given ultra k-security number s is demonstrated. If $s = n$ then $G = K_n$ is the desired graph for all $k \geq 1$.

Theorem 3.60. *For any positive integers* n, k, s *with* $n > s \geq k + 1 \geq 2$, *there exists a graph* G *of order* n *with* $s_k(G) = s_k^u(G) = s$.

3.5.6 Ultra k-security numbers of some classes of graphs

In this section, ultra k-security numbers are determined for some classes of graphs.

Theorem 3.61. *1. For any $n \geq 2$, $s_1^u(P_n) = 2$.*

2. For any $n \geq 3$, $s_1^u(C_n) = \begin{cases} 3 & \text{if } n = 3 \\ 4 & \text{if } n \geq 4 \end{cases}$.

3. For any $n \geq 2$ and $1 \leq k \leq n-2$, $s_k^u(K_n) = n$.

Theorem 3.62. *For $l \geq 2$, let $m_1 \leq m_2 \leq \ldots \leq m_l$ be positive integers with $m_l > 1$ and $\sum_{i=1}^{l} m_i = n$. Then,*

$$s_k^u(K_{m_1,m_2,\ldots,m_l}) = \begin{cases} n - \left\lfloor \dfrac{m_l - (k-1)(n-m_l)}{1+n-m_l} \right\rfloor & \text{if } 1 \leq k \leq \lfloor \frac{m_l-1}{n-m_l} \rfloor, m_l \geq \lceil \frac{n+1}{2} \rceil \\ \\ n & \text{otherwise.} \end{cases}$$

Theorem 3.63. *For integers $n \geq m \geq 3$, $s_1^u(P_m \times P_n) = 2m$.*

Theorem 3.64. *For integers $m, n \geq 4$, $s_1^u(P_m \times C_n) = min\{4m, 2n\}$ and $s_1^u(C_m \times C_n) = min\{4m, 4n\}$.*

Theorem 3.65. *For integers $m, n \geq 5$,*

$$\left\lceil \frac{12mn + 4m + 4n - 8\left\lfloor \frac{m}{5} \right\rfloor - 8\left\lfloor \frac{n}{5} \right\rfloor - 12}{13} \right\rceil \leq s_2^u(P_m \times P_n) \leq mn - \left\lfloor \frac{m}{5} \right\rfloor \left\lfloor \frac{n}{5} \right\rfloor.$$

Theorem 3.66. *For integers $m, n \geq 5$,*

$$\left\lceil \frac{12mn + 4n - 8\left\lfloor \frac{n}{5} \right\rfloor}{13} \right\rceil \leq s_2^u(P_m \times C_n) \leq mn - \left\lfloor \frac{m}{5} \right\rfloor \left\lfloor \frac{n}{5} \right\rfloor.$$

The problem of finding the ultra 2-security number of $P_m \times P_n$ is equivalent to find the maximum number of rhombuses of equal sides that can be packed inside a rectangle as shown in Figure 6. Roughly, $s_2(P_m \times C_n) = mn - x$, where x is the maximum number of disjoint rhombuses that can be packed similarly as in Figure 6 on a surface of a cylindrical square grid. Also, $s_2(C_m \times C_n) = mn - y$ where y is the maximum number of disjoint rhombuses that can be packed similarly as in Figure 6 on a surface of a torus.

Theorem 3.67. *For integers $m, n \geq 5$, $s_3^u(P_m \times P_n) = mn$.*

Similarly, it can be proved that for $m, n \geq 3$, $s_2(P_2 \times C_n) = 2n$ and $s_3^u(P_m \times C_n) = s_3^u(C_m \times C_n) = mn$.

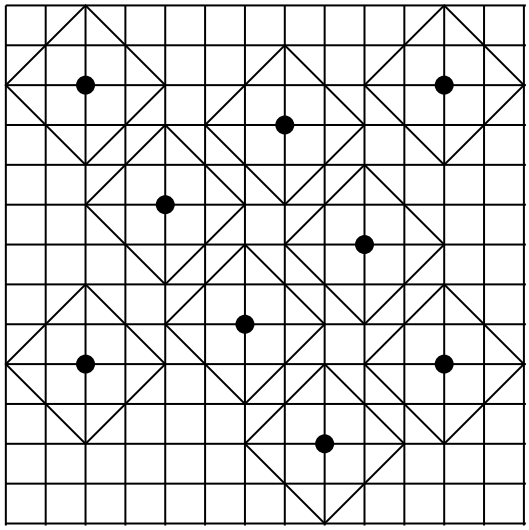

Figure 6: A minimum ultra 2-secure set S in $P_{14} \times P_{14}$. Black dots denote the vertices of $V - S$.

3.6 Chapter 5: Strong Defensive Alliances and Strong Secure Sets

For $k \geq 0$, a defensive alliance S is said to be a k-strong defensive alliance [25] if for any l with $1 \leq l \leq k$, $S - S_l$ is a defensive alliance, where S_l is any subset of S with l elements. A 0-strong defensive alliance is nothing but a defensive alliance. The minimum cardinality of a k-strong defensive alliance in a graph G is the k-strong defensive alliance number of G, denoted by $a^k(G)$. A k-strong defensive alliance S in G is said to be a minimal k-strong defensive alliance if no proper subset of S is a k-strong defensive alliance in G. The maximum cardinality of a minimal k-strong defensive alliance is the upper k-strong defensive alliance number, denoted by $A^k(G)$.

A k-strong defensive alliance S in a graph is said to be a k-strong global defensive alliance if S is also a dominating set. The k-strong global defensive alliance number $\gamma_{a^k}(G)$ and the upper k-strong global defensive alliance number $\gamma_{A^k}(G)$ are defined similar to $a^k(G)$ and $A^k(G)$.

3.6.1 Properties of k-Strong Defensive Alliances

In this section, some properties of k-strong defensive alliances are discussed, whenever it exists.

Remark 3.68. *For any $S \subseteq V$, $Att_S(v) + Def_S(v) = 1 + deg(v)\ \forall\ v \in S$, where $deg(v)$ is the degree of v.*

Proposition 3.69. *If S_1 and S_2 are k-strong defensive alliances in a graph G, then $S_1 \cup S_2$ is a k-strong defensive alliance in G.*

Theorem 3.70. *For any graph G of order at least $k+1$, $a^k(G) = k+1$ if and only if it has at least $k+1$ pendant vertices.*

Theorem 3.71. *For any graph G, $a^k(G) \geq k + \left\lceil \frac{1+\delta(G)}{2} \right\rceil$.*

Theorem 3.72. *Let S be a k-strong defensive alliance and $v \in S$ with $\deg(v) > 1$. Then $Def_S(v) \geq \left\lfloor \frac{deg(v)}{2} \right\rfloor + k + 1$.*

Theorem 3.73. *Let S be a minimal k-strong defensive alliance that has at most k pendant vertices. Then $\langle S \rangle$ is connected.*

3.6.2 Bounds for k and $a^k(G)$

Let S be a minimal k-strong defensive alliance. Then there is a subset $\{v_1, v_2, \ldots, v_{k+1}\}$ of S such that $S - \{v_1, v_2, \ldots, v_{k+1}\}$ is either empty or not a defensive alliance.

Theorem 3.74. *Let G be a graph of order $n \geq 2k+1$. If there is no vertex of degree $2k$ or $2k-1$, then $a^k(G) \leq n-1$.*

Theorem 3.75. *For any graph G, $a^k(G) = k+1$ or $a^k(G) \geq 2k$.*

Theorem 3.76. *For any graph G and $k > 1$, $a^k(G) = 2k$ if and only if $G = K_{2k}$.*

Theorem 3.77. *Let $G = (V, E)$ be a graph with l pendant vertices. For $k \geq l$, if G has a k-strong defensive alliance, then $k \leq \lfloor \frac{\triangle(G)+1}{2} \rfloor$, where $\triangle(G) = \max_{v \in V} \deg(v)$.*

Theorem 3.78. *Let $k > 0$ and G be a graph of order n with $\delta(G) \geq 2k+1$. Then $a^k(G) \leq n - \lfloor \frac{\delta(G)-(2k-1)}{2} \rfloor$.*

3.6.3 k-Strong Defensive Alliances in Some Classes of Graphs

Theorem 3.79. *For any $n \geq 3$, $a^2(W_{1,n}) = n+1$ and $W_{1,n}$ has no k-strong defensive alliance for $k \geq 3$.*

Theorem 3.80. *For $m, n > 2$ and $0 \leq k \leq min\{\lceil \frac{m+2}{2} \rceil, \lceil \frac{n+2}{2} \rceil\}$, $a^k(P_m \vee P_n) = \lfloor \frac{m+2}{2} \rfloor + \lfloor \frac{n+2}{2} \rfloor + 2k - 4$.*

Theorem 3.81. *Let $m \geq 1$, $n > 1$ be any integers. Then*

1. *$a^k(K_{m,n}) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2k$ for $m > 1$ and $k \leq min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$,*
2. *$a^k(C_m \vee \overline{K_n}) = \lfloor \frac{n+2}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 2k - 2$ for $m \geq 3$, $0 \leq k \leq \min\{\lceil \frac{n+2}{2} \rceil, \lceil \frac{m}{2} \rceil\}$.*
3. *$a^k(P_m \vee \overline{K_n}) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n+2}{2} \rfloor + 2k - 2$, $m \geq 3, 0 \leq k \leq \min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n+2}{2} \rceil\}$.*
4. *$a^k(P_2 \vee \overline{K_n}) = \lfloor \frac{n+1}{2} \rfloor + 2k$ for $n > 1$ and $0 \leq k \leq 1$.*

Theorem 3.82. *For integers $m, n > 1$, $a^1(P_m \times P_n) = \min\{2m, 2n\}$. Further $P_m \times P_n$ does not contain k-strong defensive alliance for $k \geq 2$.*

Theorem 3.83. *For any integers $m \geq 3$ and $n \geq 2$, $a^1(C_m \times P_n) = \min\{m, 2n\}$.*

Theorem 3.84. *For integers $m, n \geq 3$, $a^1(C_m \times C_n) = \min\{2m, 2n\}$.*

Theorem 3.85. *For $m, n \geq 3$, $a^2(C_m \times P_n) = mn$. Further there is no k-strong defensive alliance in $C_m \times P_n$ for $k \geq 3$.*

Theorem 3.86. *For $m \geq 3$ and $n > 1$, $a^2(C_m \times C_n) = mn$. Further there is no k-strong defensive alliance in $C_m \times C_n$ for $k \geq 3$.*

3.6.4 k-Strong Secure Sets

Similar to k-strong defensive alliances, a k-strong secure set is defined as follows.

Definition 28. *For an integer $k \geq 0$, a secure set S is said to be a k-strong secure set if for any l with $1 \leq l \leq k$, $S - S_l$ is a secure set, where S_l is any subset of S with l elements.*

For a graph G and for an integer $k \geq 0$, the minimum cardinality of a k-strong secure set is the k-strong security number $s^k(G)$ of G, whenever exists. It is easily seen that $V(G)$ is a 1-strong secure set. Thus $s^1(G)$ exists for every graph G.

Proposition 3.87. *Let G be a graph and $k \geq 0$ be an integer. Then,*

1. $s^{k+1}(G) \geq s^k(G) \geq s(G)$.
2. $s^k(G) \geq k + 1$.
3. $s^k(G) \geq 2k$ *whenever* $s^k(G) > k + 1$.

4 List of Publications/Papers Presented/Communicated

4.1 Articles Published/Accepted/Communicated

1. Karthik K., Chandru Hegde and B. Sooryanarayana, k-distance secure sets in graphs, Materials Today: Proceedings (Accepted), 2023.
   https://doi.org/10.1016/j.matpr.2023.05.382

2. K Karthik and Chandru Hegde, k-Secure Sets and k-Security Number of a Graph, Communications in Combinatorics and Optimization (Accepted), 2023.
   https://doi.org/10.22049/cco.2023.28048.1427

3. Karthik K., Chandru Hegde and B. Sooryanarayana, k-Strong Defensive Alliances in Graphs, South East Asian J. of Mathematics and Mathematical Sciences Vol. 19, No. 1 (2023), pp. 201-210.
   https://doi.org/10.56827/SEAJMMS.2023.1901.16

4. K Karthik and Chandru Hegde, On Generalized Distance Secure Sets in Graphs, Proc. Jangjeon Math. Soc. 27 (2024), No. 4, pp. 615 - 622.
   http://dx.doi.org/10.17777/pjms2024.27.4.615

5. Karthik K and Chandru Hegde, On Ultra k-Security Number of a Graph, In Graphs and Combinatorics (Proceedings of International Conference on Graphs and Combinatorics-2022), pp. 117-122. ISBN: 978-93-95283-23-6.

6. K Karthik and Chandru Hegde, On Ultra k-Secure Sets in Graphs (Under Review).

4.2 Papers presented in International/National conferences

1. Presented a paper titled 'Some Bounds on k-Strong Defensive Alliance Number of a Graph' in National Seminar on Computational Mathematics and Research Culture held at St. Agnes College Mangaluru on 26/05/2022.

2. Presented a paper titled 'k-Secure Sets in Graphs' in 18^{th} Annual Conference of ADMA (Academy of Discrete Mathematics and Applications) held at Vellalar College for Women, Erode, Tamilnadu during 09-11, June 2022.

3. Presented a paper titled 'Connected k-Distance Secure Sets in Graphs' in 'National Conference on Scope of Research in Discrete Mathematics as per NEP' at Poornaprajna College, Udupi during 28-30, November 2022.

4. Presented a paper titled 'On Ultra k-Security Number of a Graph' in International Conference on Graphs and Combinatorics held at Mangalore University during 01-03, December 2022.

5. Presented a paper titled 'k-Distance Secure Sets in Graphs' in the conference 'Materials Sciences and Mathematics for Advanced Technology-2022' held at NMAM Institute of Technology, Nitte during 22-23, December 2022.

6. Presented a paper 'On Generalized Distance Secure Sets in Graphs' in 'International Conference on Number Theory and Graph Theory-2023' held at Manipal Institute of Technology, Manipal during 18-20, January 2023.

7. Presented a paper titled 'On secure partition number of a graph' and got best paper award in 20^{th} Annual conference of ADMA 'International Conference on Discrete Mathematics-2024' held at Savitribai Phule Pune University during 07-10, June 2024.

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